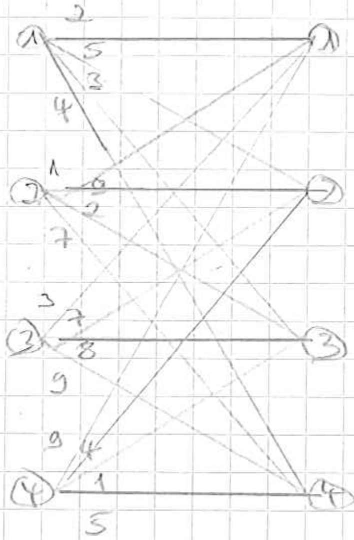
3.4 The Assignment Problem

3.4.1. Ex (Assignment or Perfect Bipartite Matching Problem)



$G=(V\,\dot\cup\, U, E)$ complete bipartite graph

$|V|=|U|$ equal sized sides

$c_{ij}\geq 0$ costs → Ex

$M\subseteq E : |M\cap\delta(i)|=1 \ \forall i\in V\cup U$ perfect matching

$c(M)\to\min$ min. cost matching

$$\begin{array}{c|cccc} 2 & 2 & 5 & 3 & 4\\ 1 & 1 & 9 & 2 & 7\\ 3 & 3 & 7 & 8 & 9\\ 1 & 9 & 4 & 1 & 5 \end{array} = (c_{ij}) \text{ cost matrix}$$

assignment (example)

row minima: $u_i=\min_j c_{ij}$ row minima

$c(M)\geq 2+1+3+1=7 \ \forall M$

column minima

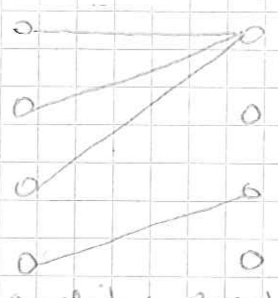
$$\begin{array}{cccc} 0 & 3 & 0 & 2\\ \hline 0 & 3 & 1 & 2\\ 0 & 7 & 1 & 6\\ 0 & 4 & 5 & 6\\ 8 & 3 & 0 & 4 \end{array}$$

$c(M)\geq (0+3+0+2)+7=12$

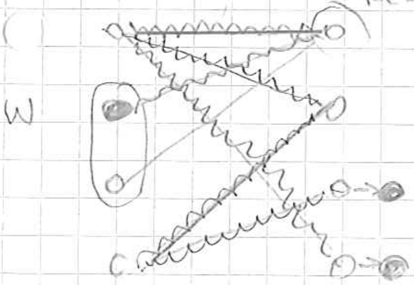
$=(\bar c_{ij})=(c_{ij}-u_i)$ reduced cost matrix

$\bar E=\{e\in E : \bar c_e=0\}$

$\bar G=(U,V,\bar E)$ equality graph



equality graph



$$\begin{array}{c|cccc} & 0 & 3 & 0 & 2\\ \hline N(W)\ 2 & 0 & 0 & 1 & 0\\ 1 & 0 & 4 & 1 & 4\\ 3 & 0 & 1 & 5 & 4\\ 1 & 8 & 0 & 0 & 2 \end{array}$$

$|N(W)|<|W|\to$ no perfect matching of 0-entries

$v_j=\min_i c_{ij}$ col. minima

$=(\bar c_{ij})=(c_{ij}-u_i-v_j)$

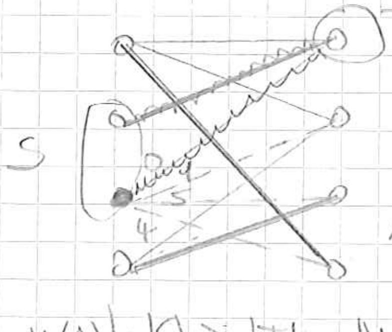
□ matching (any)

● $v\in U\setminus M$ unmatched node

⇝ A alternating tree reaches unmatched node

$S=U\cap V(A)$, $T=V\cap V(A)$ visited nodes

△ larger matching



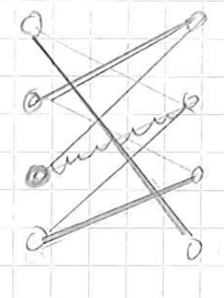
$$\begin{array}{c|cccc} & 0-1 & 3 & 0 & 2\\ \hline 2 & 0 & 0 & 1 & 0\\ 1+1 & 0 & 4 & 1 & 4\\ 1+3 & 0 & 1 & 5 & 4\\ 1 & 8 & 0 & 0 & 2 \end{array}$$

$|U\cap V(A)|=|S|>|T|=|V\cap V(A)|$

$$\varepsilon=\min_{\substack{i\in\bullet\\ j\in V}} c_{ij}$$

$u_i \mathrel{+}= \varepsilon \ \forall i\in S$

$v_j \mathrel{-}= \varepsilon \ \forall j\in T$



$$\begin{array}{c|cccc} & -1 & 3 & 0 & 2 \\ \hline 2 & 1 & 0 & 1 & \boxed{0} \\ 2 & \boxed{0} & 3 & 0 & 3 \\ 4 & 0 & 0 & 4 & 3 \\ 1 & 9 & 0 & \boxed{0} & 2 \end{array} = (\bar{c}_{ij}) \quad \text{optimal}$$

$c(M) \geq 9 + 4 = 13$

$+$

3.4.2 Def. (Assignment or Perfect Bipartite Matching Problem)

Input: $G = (U \dot\cup V, E)$, $E = U \times V$, $c_{ij} \in \mathbb{R} \; \forall ij \in E$, $|U| = |V|$

Output: $M \subseteq E$ perfect matching of min cost $c(M)$

3.4.3 Def. (Assignment Terminology):

i) Let $F \subseteq E$

$U(F) := \{ i \in U : \exists j \in V : ij \in F \}$, $V(F) := \{ j \in V : \exists i \in U : ij \in F \}$

ii) Let $u \in \mathbb{R}^U$, $v \in \mathbb{R}^V$

$\bar{c}_{ij} := \bar{c}_{ij}(u,v) = c_{ij} - u_i - v_j \;\; \forall ij \in E$ reduced cost w.r.t. u, v

$\bar{E} := \{ ij \in E : \bar{c}_{ij} = 0 \}$, $\bar{G} := (U \dot\cup V, \bar{E})$ equality graph

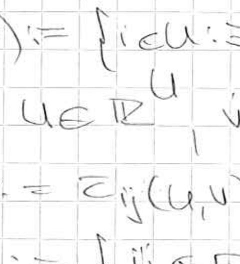
iii) Let $u \in \mathbb{R}^U$, $v \in \mathbb{R}^V$, $M \subseteq \bar{E}$ matching, $r \in U \setminus U(M)$, $P \subseteq \bar{E}$ path, $r \in U(P)$

P alternating rj-path $:\Leftrightarrow$ $P = \overset{r}{u_0} v_1, \underset{M}{v_1 u_2}, u_2 v_3, \ldots$ alternates edges $\notin M / \in M$

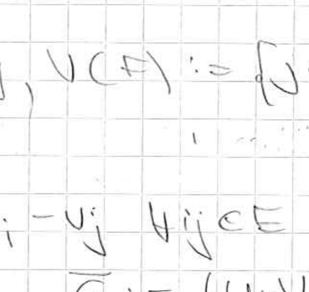
P augmenting rj-path $:\Leftrightarrow$ $P = r \ldots j$ alternating, $j \in V \setminus V(M)$

iv) Let $u \in \mathbb{R}^U$, $v \in \mathbb{R}^V$, $M \subseteq \bar{E}$ matching, $r \in U \setminus U(M)$, $T \subseteq \bar{E}$ tree, $r \in U(T)$

T alternating r-tree $:\Leftrightarrow$ all paths in T are alternating



augmenting



no augmenting path $\Rightarrow |U(T)| > |V(T)|$

v) $u \in \mathbb{R}^U$, $v \in \mathbb{R}^V$ dual feasible $:\Leftrightarrow$ $\bar{c}_{ij} = c_{ij} - u_i - v_j \geq 0$

3.4.4 Prop. (IP-Formulation of the Assignment Problem)

The assignment problem can be formulated as an IP as follows:

$$\text{(IP)} \quad \min \sum_{ij \in E} c_{ij} x_{ij} \qquad \Leftrightarrow: \quad \min c^T x$$

$$\sum_{j \in V} x_{ij} = 1 \quad \forall i \in U \qquad\qquad A(G) x = \mathbb{1}$$

$$\sum_{i \in U} x_{ij} = 1 \quad \forall j \in V \qquad\qquad x \in \{0,1\}^E$$

$$x_{ij} \in \{0,1\} \quad \forall ij \in E$$

3.4.5 Prop. (Total unimodularity): $A(G)$ is TU.

Proof: Ex. □

3.4.6 Cor. (LP-Formulation of the Assignment Problem): The LP-relaxation of the assignment problem has 0/1-extreme solutions

$$(LP)\ \min \sum c_{ij} x_{ij} \quad = (DP)\ \max \sum_i u_i + \sum_j v_j$$
$$\sum_j x_{ij} = 1 \quad \forall i\ (u_i) \qquad u_i + v_j \le c_{ij}\ \forall ij$$
$$\sum_i x_{ij} = 1 \quad \forall j\ (v_j)$$
$$x_{ij} \ge 0 \quad \forall ij$$

3.4.7 Cor. (Dual Solution): Let $u \in \mathbb{R}^U$, $v \in \mathbb{R}^V$ be dual feasible. Then $\sum u_i + \sum v_j \le c(M)$ $\forall$ matching M.

Proof: $c(M) = \sum_{ij \in M} c_{ij} = \sum_{i \in U} c_{ij(i)} \ge \sum_{i \in U} u_i + v_{j(i)} = \sum_{i \in U} u_i + \sum_{j \in V} v_j$. □

3.4.8 Prop. (Alternating Paths): Let $M \subseteq \bar{E}$ be a matching. Then M is maximal (w.r.t. inclusion) $\Leftrightarrow$ $\nexists$ augmenting path in $\bar{G}$ $\Leftrightarrow$ $\nexists$ augmenting path in a maximal forest of alternating trees.

Proof: Ex. □

3.4.9 Alg. (Hungarian method)

Input: $G = (U \dot\cup V, E)$, $E = U \times V$, $c_{ij} \ge 0$ $\forall ij \in E$, $|U| = |V|$

Output: $M \in \operatorname{argmin}_{M \text{ perfect matching}} c(M)$, $(u,v) \in \mathbb{R}^{U \times V} \in \operatorname{argmax} DP$ $\Big\} \Rightarrow c(M) = \sum u_i + \sum v_j$

Data structures: $\bar{G}$, $\bar{c}$

1. $u \leftarrow (\min_{j \in V} c_{ij})_{i \in U}$, $v \leftarrow (\min_{i \in U} c_{ij} - u_i)_{j \in V}$
2. $M \leftarrow$ some maximal matching in $\bar{G}$
3. If $|M| = |U|$ then stop, output M, (u,v)
4. $T \leftarrow$ maximal alternating r-tree

4a. if $\exists$ alternating v_j-path $p \subseteq T$ then

$M \leftarrow (M \setminus (p \cap M)) \cup (p \setminus M)$, goto 3 // primal update

4b. else

$\varepsilon \leftarrow \min\limits_{\substack{i \in U(T) \\ j \notin V(T)}} \bar{c}_{ij}$ // dual update

4c. $u_i \leftarrow u_i + \varepsilon \quad \forall i \in U(T)$

4d. $v_j \leftarrow v_j - \varepsilon \quad \forall j \in V(T)$

goto 4

endif

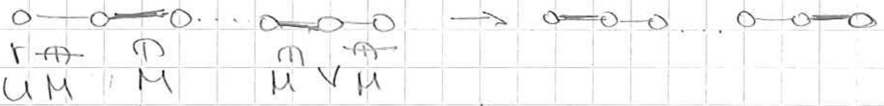
3.4.10 Thm (Kuhn [1960]): Alg. 3.4.9 is correct.

Proof:

i) Claim: w.l.o.g. c_{ij} integer $\forall ij$ ✓

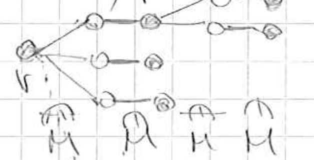
ii) Claim: M grows by 1 edge in 4a

Proof:



iii) Claim: $|U(T)| > |V(T)|$ in 4b

Proof:



$|U(T)| > |V(T)|$

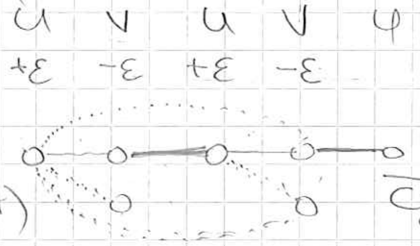
$|\{\bullet\}| > |\{\circ\}|$

iv) Claim: $\varepsilon \geq 1$

Proof: $T \subseteq \bar{E}$, $\varepsilon = \min\limits_{ij \in \delta(U(T)) \cap (E \setminus \bar{E})} \bar{c}_{ij} > 0$!

v) Claim: (u_i, v_i) dual feasible

Proof:



$\bar{c}_{ij} \geq \varepsilon$

vi) Claim: If Alg. 3.4.9 terminates in 3, M is optimal

Proof: $\bar{c}(M) = 0 \iff \sum\limits_{ij \in M} \bar{c}_{ij} = \sum\limits_{ij \in M} (c_{ij} - u_i - v_j) = 0$

$\iff \sum\limits_{ij \in M} c_{ij} = \sum u_i + \sum v_j \leq c(M') \; \forall M'$

vii) Claim: Alg. 3.4.9 terminates finitely.

Proof: ii), iv). □

3.4.11 Prop (Run time of the Hungarian method): Alg. 3.4.9 runs in $O(|V|^3)$ time. Proof. Ex. □

3.5 Lagrangean Relaxation

3.5.1 Rem (Compound Optimization Problems): Consider

(P) $\min c^T x,\ \underbrace{Dx = d}_{\text{complicated (C)}},\ \underbrace{Ax = b,\ x \geq 0}_{\text{tractable (T)}}$

How to reduce (P)(C,T) to (P)(T) / get rid off (C)?

3.5.2 Def (Lagrange Relaxation): Let $D \in \mathbb{R}^{m \times n}$, $\underbrace{X}_{\text{e.g. } Ax \leq b,\ x \in \{0,1\}^n} \subseteq \mathbb{R}^n$ closed,

(P) $\min c^T x,\ Dx = d,\ x \in X$. Then

$(L_{Dx=d})$ $\min c^T x,\ (c^T - \lambda^T D)x + \lambda^T d,\ x \in X$

is the Lagrange relaxation of (P) w.r.t. $Dx = d$.

$f: \mathbb{R}^m \to \mathbb{R},\ \lambda \mapsto f(\lambda) := \min\limits_{x \in X} (c^T - \lambda^T D)x$

Lagrange function of (P) w.r.t. $Dx = d$;

$f(\lambda) = \min\limits_{x \in X} (c^T - \lambda^T D)x + \lambda^T d$

Subproblem of (L).

3.5.3 Thm (Elementary properties of a Lagrange relaxation, Geoffrion [ ]): Let $v(P) := \min c^T x,\ Dx = d,\ x \in X \in \mathbb{R}_\infty$.

a) $\max f(\lambda) \leq v(P)$

b) Let $X = \{Ax = b, x \geq 0\}$, $X \cap \{Dx = d\} \neq \emptyset$. Then

$\max f(\lambda) = v(P)$,

in particular, feasible LPs and their Lagrange relaxations have the same optimal objective values.

c) Let X be $\left\{ \begin{matrix} \text{finite} \\ \text{a polytope} \end{matrix} \right\}$ and $X \cap \{Dx = d\} \neq \emptyset$. Then f is

i) concave, ii) piecewise affine, iii) bounded from above.

Proof:

a) $f(\lambda) = \min\limits_{x \in X} \underbrace{(c^T - \lambda^T D)x + \lambda^T d}_{= c^T x - \lambda^T (Dx - d)} \leq \min\limits_{\substack{x \in X \\ Dx = d}} c^T x - \lambda^T \underbrace{(Dx - d)}_{=0} = v(P)$.

b) $\min\limits_{\substack{Ax = b \\ Dx = d \\ x \geq 0}} c^T x \overset{\text{duality}}{\underset{\text{Thm}}{=}} \max\limits_{\mu^T A + \lambda^T D \leq c^T} \mu^T b + \lambda^T d = \max\limits_{\lambda} \lambda^T d + \max\limits_{\mu^T A \leq c^T - \lambda^T D} \mu^T b$

$\max\limits_{\lambda} f(\lambda) = \max\limits_{\lambda} \min\limits_{\substack{Ax = b \\ x \geq 0}} (c^T - \lambda^T D)x + \lambda^T d = \max\limits_{\lambda} \lambda^T d + \min\limits_{\substack{Ax = b \\ x \geq 0}} (c^T - \lambda^T D)x$

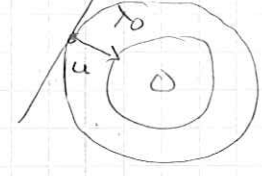
c) Let $X = \begin{Bmatrix}\{x_1,\dots,x_k\}\\ \mathrm{conv}\{x_1,\dots,x_k\}\end{Bmatrix}$. Then

$$f(\lambda) = \min_{x\in X} (c^T-\lambda^T D)x+\lambda^T d = \min_{i=1}^{k} \underbrace{c^T x_i - \lambda^T(Dx_i-d)}_{\text{affine in } \lambda} \le v(P) \underbrace{< \infty}_{X\cap\{Dx=d\}\neq\emptyset}$$

3.5.4 Def. (Subgradient, subdifferential): Let $f: \mathbb{R}^m \to \mathbb{R}$ be concave.

a) $u \in \mathbb{R}^m$ s.t. $f(\lambda) \le f(\lambda_0) + u^T(\lambda-\lambda_0)\ \forall \lambda$ subgradient of f at λ_0

b) $\partial f(\lambda_0) = \{u\in\mathbb{R}^m : u \text{ subgradient of } f \text{ at } \lambda_0\}$
subdifferential of f at λ_0.



3.5.5 Prop. (Diff'able case): Let $f: \mathbb{R}^m \to \mathbb{R}$ be concave and diff'able at λ_0. Then

$$\partial f(\lambda_0) = \{f'(\lambda_0)\}$$

Proof: Ex. $\square$

3.5.6 Prop (Polyhedral case): Let $X = \begin{Bmatrix}\{x_1,\dots,x_k\}\\ \mathrm{conv}\{x_1,\dots,x_k\}\end{Bmatrix} \neq \emptyset$.

$$\partial f(\lambda_0) = \mathrm{conv}\{-(Dx_i-d) : x_i \in \operatorname{argmin} f(\lambda_0)\}$$

Proof: Let $X(\lambda) := \operatorname{argmin} f(\lambda_0)$, $\lambda\in\mathbb{R}^m$, $u_i := -(Dx_i-d)$.

"$\supseteq$": $\forall x_j \in X(\lambda_0)$:

$$f(\lambda_0) + u_j^T(\lambda-\lambda_0) = \underbrace{c^T x_j - \cancel{\lambda_0^T(Dx_j-d)}}_{=f(\lambda_0)} \underbrace{-(Dx_j-d)^T}_{=u_j}(\lambda - \cancel{\lambda_0})$$
$$\ge \min_{i=1}^{k} c^T x_i - \lambda^T(Dx_i-d)$$
$$= f(\lambda)$$

"$\subseteq$": $\min_{x_i\notin X(\lambda_0)} c^T x_i - \lambda_0^T(Dx_i-d) - f(\lambda_0) > 0$

$\Rightarrow X(\lambda) \subseteq X(\lambda_0)\ \forall \lambda \in U_\varepsilon(\lambda_0)$ for some $\varepsilon > 0$

Let $u \notin \mathrm{conv}\{u_i : x_i\in X(\lambda_0)\}$.

separating hyperplane $\Rightarrow$ $\exists \pi\in\mathbb{R}^m$: $u^T\pi < u_i^T\pi\ \forall x_i\in X(\lambda_0)$

Then $\Rightarrow$
$$f(\underbrace{\lambda_0+\varepsilon\pi}_{=\lambda}) = \min_{x_i\in X(\lambda_0)} c^T x_i \underbrace{- (\lambda_0+\varepsilon\pi)^T(Dx_i-d)}_{= -\lambda_0^T(Dx_i-d) + \varepsilon\pi^T u_i}$$
$$= f(\lambda_0) + \min_{x_i\in X(\lambda_0)} \varepsilon\pi^T u_i$$
$$> f(\lambda_0) + \varepsilon\pi^T u$$
$$= f(\lambda_0) + u^T(\underbrace{\lambda_0+\varepsilon\pi}_{=\lambda} - \lambda_0) \ \lightning \quad \square$$

3.5.7 Alg. (Subgradient method)

Input: $f : \mathbb{R}^n \to \mathbb{R}$ concave

$\lambda_0 \in \mathbb{R}^n$ starting point

$(\alpha_k)_{k=1}^{\infty}$, $\alpha_k > 0$ step lengths

Output: $(\lambda_k)_{k=1}^{\infty}$

1. $k \leftarrow 0$, $u_0 \leftarrow u \in \partial f(\lambda_0)$
2. $\lambda_{k+1} \leftarrow \lambda_k + \alpha_k u_k$

   $u_{k+1} \leftarrow u \in \partial f(\lambda_{k+1})$
3. goto 2.

3.5.8 Thm (Convergence of Subgradient method): Let $f : \mathbb{R}^n \to \mathbb{R}$ be concave and

i) $f^* = \max f < \infty$

ii) $\|u\|_2 \leq L \quad \forall u \in \partial f$

iii) $\sum \alpha_k^2 < \infty$, $\sum_k \alpha_k \to \infty$

Then $\lim_{k \to \infty} \max_{j=0}^{k} f(\lambda_j) = f^*$.

Proof: Let $\lambda^* \in \arg\max f$.

$$\|\lambda_{k+1} - \lambda^*\|_2^2 = \|\lambda_k + \alpha_k u_k - \lambda^*\|_2^2$$

$$= \|\lambda_k - \lambda^*\|_2^2 + 2\alpha_k \underbrace{u_k^T(\lambda_k - \lambda^*)}_{\leq f(\lambda_k) - f^*} + \alpha_k^2 \|u_k\|_2^2$$

$$\leq \|\lambda_0 - \lambda^*\|^2 - \sum_{j=0}^{k} 2\alpha_k \left[f^* - f(\lambda_k)\right] + \sum_{j=0}^{k} \alpha_j^2 \|u_j\|_2^2$$

$$\Rightarrow \cancel{\underbrace{\|\lambda_{k+1} - \lambda^*\|_2^2}_{\leq 0}} + \sum_{j=0}^{k} 2\alpha_k \left[f^* - f(\lambda_k)\right] \leq \|\lambda_0 - \lambda^*\|_2^2 + \sum \alpha_j^2 L^2$$

$$\Rightarrow f^* - \max_{j=0}^{k} f(\lambda_k) \leq \Big[\underbrace{\|\lambda_0 - \lambda^*\|_2^2}_{\equiv C} + \underbrace{\sum_{j=0}^{k} \alpha_j^2 L^2}_{\leq C}\Big] \Big/ \underbrace{2\sum_{j=0}^{k} \alpha_j}_{\to \infty} \xrightarrow[k \to \infty]{} 0. \quad \square$$

3.5.9 Ex (Capacitated Facility Location Problem):

(CFLP) $\min \sum d_{ij} x_{ij} + \sum_i f_i y_i$

(D) $\sum_i x_{ij} \geq 1 \quad \forall j \in J$ — demand constraints

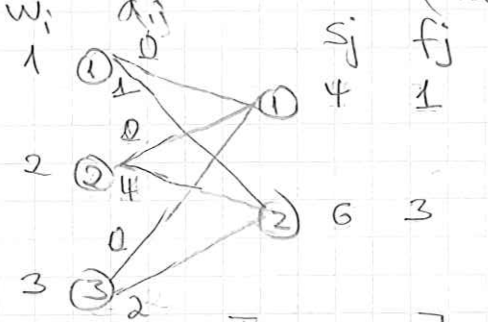
(B) $y_i \geq x_{ij} \quad \forall ij \in E$ — (variable) upper bounds

(C) $s_i y_i \geq \sum_j w_j x_{ij} \quad \forall i \in I$ — capacity constraints

(N) (I) $0 \leq x, y \leq \mathbb{1}$, $y_i \in \mathbb{Z}$ $\forall i \in I$ — non-negativity, integrality

$$(L_c) \quad \max_{\lambda \in \mathbb{R}^J_{\geq 0}} \min \sum d_{ij} x_{ij} + \sum f_i y_i - \sum \lambda_j \underbrace{\Big(s_j y_i - \sum_j w_j x_{ij}\Big)}_{\substack{\text{feas} \Leftrightarrow \geq 0 \Rightarrow \lambda_j = 0 \\ \text{infeas} \Leftrightarrow < 0 \Rightarrow \lambda_j \to \infty}}$$

$$\underbrace{\begin{array}{lll} (D) & \sum_i x_{ij} \geq 1 & \forall j \in J \\ (B) & y_i \geq x_{ij} & \forall ij \in E \\ (N) & 0 \leq x, y \leq \mathbb{1} & \\ (I) & y_j \in \mathbb{Z} & \forall i \in I \end{array}}_{\text{(MUFL) with modified objective}}$$



optimal solution: $x^*_{11} = x^*_{21} = 1$, $x^*_{31} = \frac{1}{3}$, $x^*_{32} = \frac{2}{3}$, cost $\frac{4}{3} + 4 = \frac{16}{3}$

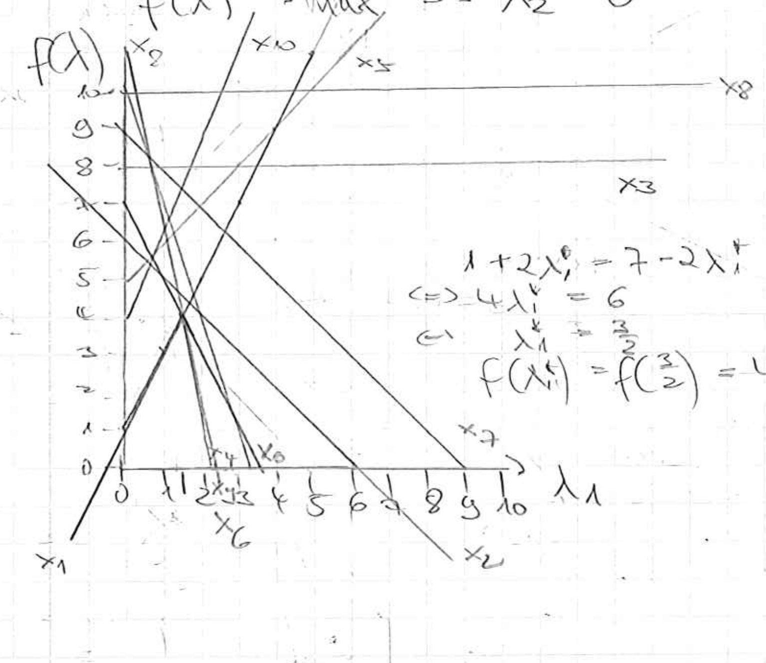
optimal LP solution: $y^*_1 = 1$, $y^*_2 = \frac{2}{3}$

cost: $\frac{4}{3} + 1 + 2 = \frac{4}{3} + 3 = \frac{13}{3}$

$$d^T x + f^T y - \lambda_1 \underbrace{c_1}_{= s_1 y_1 - \sum w_i x_{i1}} - \lambda_2 \underbrace{c_2}_{= s_2 y_2 - \sum w_i x_{i2}} \qquad f(\lambda)$$

	I: 1	I: 2	I: 3	J: 1	J: 2	$d^T x + f^T y - \lambda_1 c_1 - \lambda_2 c_2$
✓ x_1	1	1	1	1	0	$0 + 1 - \lambda_1(-2) - \lambda_2(0) = 1 + 2\lambda_1$
✓ x_2	1	1	2	1	1	$2 + 4 - \lambda_1(1) - \lambda_2(3) = 6 - \lambda_1 - 3\lambda_2$
✓ x_3	1	2	1	1	1	$4 + 4 - \lambda_1(0) - \lambda_2(4) = 8$
✓ x_4	1	2	2	1	1	$6 + 4 - \lambda_1(3) - \lambda_2(1) = 10 - 3\lambda_1 - \lambda_2$
✓ x_5	2	1	1	1	1	$1 + 4 - \lambda_1(-1) - \lambda_2(5) = 5 + \lambda_1 - 5\lambda_2$
✓ x_6	2	1	2	1	1	$3 + 4 - \lambda_1(2) - \lambda_2(2) = 7 - 2\lambda_1 - 2\lambda_2$
✓ x_7	2	2	1	1	1	$5 + 4 - \lambda_1(1) - \lambda_2(3) = 9 - \lambda_1 - 3\lambda_2$
✓ x_8	2	2	2	0	1	$7 + 3 - \lambda_1(0) - \lambda_2(0) = 10$
x_9	2	2	2	1	1	$7 + 4 - \lambda_1(4) - \lambda_2(0) = 11 - 4\lambda_1$
x_{10}	1	1	1	1	1	$0 + 4 - \lambda_1(-2) - \lambda_2(6) = 4 + 2\lambda_1 - 6\lambda_2$

$f(\lambda) \to \max \Rightarrow \lambda_2 = 0$



$1 + 2\lambda^*_1 = 7 - 2\lambda^*_1$
$\Leftrightarrow 4\lambda^*_1 = 6$
$\Leftrightarrow \lambda^*_1 = \frac{3}{2}$

$f(\lambda^*_1) = f(\frac{3}{2}) = 4$

optimal LP solution:

$y^*_1 = 1$, $y^*_2 = \frac{1}{2}$

$x^*_{11} = x^*_{31} = \frac{1}{2}$, $x^*_{21} = 1$

$x^*_{12} = x^*_{32} = \frac{1}{2}$

cost $1 + \frac{3}{2} + \frac{3}{2} = 4$!

3.5.10 Def (Capacitated Facility Location Problem revisited):

Let I, J be finite sets, $f \in \mathbb{R}^I$, $s \in \mathbb{R}^I_{>0}$, $w \in \mathbb{R}^J_{>0}$, $d_{ij} \in \mathbb{R}^{I \times J}$.

(CFLP) $\min \sum d_{ij} x_{ij} + \sum f_i y_i = Z$ — capacitated facility location problem

$\to$ (D) $\sum_i x_{ij} \geq 1 \quad \forall j \in J$ — demand

(B) $y_i \geq x_{ij} \quad \forall ij \in I \times J$ — variable upper bounds

$\to$ (C) $s_i y_i \geq \sum_j w_j x_{ij} \quad \forall i \in I$ — capacities

/ (N) $0 \leq x, y \leq 1$ — non-negativity

* (I) $y_i \in \mathbb{Z} \quad \forall i \in I$ — integrality

(T) $\sum s_i y_i \geq \sum w_j$ — total demand

For $R, L \subseteq \{D, B, C, N, I, T\} =: S$, $R \cap L = \emptyset$, $\overline{R \cup L} = S \setminus (R \cup L)$ let

$Z_L^R := Z_L(\overline{R \cup L})$

i.e., (Z_L^R) arises from (Z) by $\left.\begin{matrix}\text{dropping} \\ \text{a Lagrangean relaxation of}\end{matrix}\right\}$ constraints $\begin{Bmatrix} R \\ L \end{Bmatrix}$

We will consider all possible relaxations

$B, T \in R, L, \overline{RL}$, $D, C \in L, \overline{RUL}$. $3 \times 3 \times 2 \times 2$ relaxations

$B, T \in R, \overline{RL}$, $I \in R$ $\quad 2 \times 2$ (LP) "

| $N \in R, L$ has no computational gain

$D, C \in R$ looses essential properties

$I \in R \underset{\text{LP-relaxation}}{\Rightarrow} X \in L \Leftrightarrow X \in \overline{LUR} \quad \forall X \in \{DBCNT\}$.

3.5.11 Obs: Let $(A_i x \leq b_i) =: S_i$, $i = 1, \ldots, k$ be ineq. systems,

$P(S_1 \ldots S_k) = \{ \quad A_1 x \leq b_1, \ldots, A_k x \leq b_k\}$

$\operatorname{conv}(S_1 \ldots S_k I) = \operatorname{conv}\{x \in \mathbb{Z} : A_1 x \leq b, \ldots, A_k x \leq b_k\}$

a) $P(S_1 \ldots S_k) = \bigcap_{i=1}^k P(S_i)$

b) $\operatorname{conv}(S_1 S_2 I) \subseteq P(S_1) \cap \operatorname{conv}(S_2 I) \subseteq P(S_1 S_2)$.

Proof: a) By definition. b) ✓ □

3.5.12 Thm (Cornuéjols, Sridharan & Thizy [1991]):

(i) $Z^{BI} \leq Z^I \leq Z_C^T \leq Z_C \leq Z$

(ii) $Z^I \leq Z_D \leq Z_C \quad (Z^I \leq Z_{D1} \leq Z_D \leq Z_C)$

(iii) $Z^{BI} \leq Z_C^B \leq Z_D$.

(iv) $z = z_B = z^B = z_T = z^T = z_{TB} = z_T^B = z_B^T = z^{TB}$ ✓

(v) $z_D = z_{DC} = z_{BD} = z_{BC} = z_{BDC} = z_D^B$ ✓

(vi) $z_C^B = z_{DC}^B$

(vii) $z_C^T = z_{TC}$

(viii) $z^I = z^{TI} = z_{TD} = z_D^T = z_{TDC} = z_{DC}^T = z_{BTC} = z_{BC}^T$

$= z_{BTDC} = z_{BDC}^T = z_{BTC} = z_{BD}^T = z_{TD}^B = z_D^{BT}$

(ix) $z^{BI} = z^{TBI} = z_{TC}^B = z_C^{TB} = z_{TDC}^B = z_{DC}^{TB}$.

Proof:

i) $z^{BI} \overset{✓}{\le} z^I \overset{(✓)}{\le} z_C^T \overset{✓}{\le} z_C \overset{✓}{\le} z$

$z^I \overset{(viii)}{=} z^{IT} = z_C^T$

ii) $z^I \overset{✓}{\le} z_D \overset{(✓)}{\le} z_C$

$z^I = z_D^I$, $z_D \overset{(v)}{=} z_{DC}$

iii) $z^{BI} \overset{✓}{\le} z_C^B \overset{(✓)}{\le} z_D$

$z^{BI} = z_C^{BI}$, $z_C^B \overset{(vi)}{=} z_{DC}^B$

iv) Claim: $\mathrm{conv}(DC \mid N) = \mathrm{conv}(TBDC \mid N)$.

Proof: $\sum_{i \in I} (C) \Leftrightarrow \sum s_i y_i \ge \sum_{ij} w_j x_{ij} = \sum w_j \sum x_{ij} \overset{(D)}{\ge} \sum w_j$.

$(C), (I) \Rightarrow (T)$. □

v) α) $z_D = z_{BDC}$ $(= z_{DC} = z_{BD} = z_{BC})$

$z_D = \min d^T x + f^T y,\ (x,y) \in P(D) \cap \underbrace{\mathrm{conv}(BC \mid NT)}_{=\{Ax \le b\}}$

$z_{BDC} = \min d^T x + f^T y,\ (x,y) \in \underbrace{P(BDC)}_{= P(D) \cap P(BC)} \cap \mathrm{conv}(INT)$.

Claim: $P(BC) \cap \mathrm{conv}(INT) = \mathrm{conv}(BC \mid NT)$.

Proof: "$\supseteq$" by 3.5.11 b).

"$\subseteq$": Let $(\bar{x},\bar{y})$ be an extreme point of $P(BC) \cap \mathrm{conv}(INT)$.

$(\bar{x},\bar{y}) \in \{0,1\}^{|I \times J|} \Rightarrow (\bar{x},\bar{y}) \in \mathrm{conv}(BC \mid NT)$. Otherwise

$\bar{y} \in \underbrace{\mathrm{conv}(INT)}_{\text{integer polytope}}|_y \Rightarrow \bar{y} = \frac{1}{2} y_1 + \frac{1}{2} y_2,\ y_1, y_2 \in \mathrm{conv}(INT),\ y_1 \neq y_2$

Set $x_{ij}^k := \begin{cases} 0, & \bar{y}_i = 0 \\ \bar{x}_{ij} \cdot y_i^k / \bar{y}_i, & \bar{y}_i \neq 0 \end{cases}$, $k = 1,2 \Rightarrow \bar{x} = \frac{1}{2} x_1 + \frac{1}{2} x_2$,

$(x_k, y_k) \in P(BC) \cap \mathrm{conv}(INT)$, $k = 1,2 \Rightarrow (\bar{x},\bar{y})$ ¬ extreme. ↯ □

$x_{ij}^k = \bar{x}_{ij} \frac{y_i^k}{\bar{y}_i} \le 1$

$\Leftrightarrow \bar{x}_{ij} \cdot \underbrace{y_i^k}_{\le 1} \le \bar{y}_i$ ✓

$x_{ij}^k = \bar{x}_{ij} \frac{y_i^k}{\bar{y}_i} \le y_i^k$ ✓

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$\beta)\ z_D = z_D^B$

Claim: $P(D) \cap \mathrm{conv}(BCINT) = P(D) \cap \mathrm{conv}(CINT)$

Proof: $(C,I,N) \Rightarrow (B)$. $\square$

vi) Claim: $P(\cancel{C}) \cap \mathrm{conv}(DINT) = \underbrace{P(DC)}_{=P(D)\cap P(C)} \cap \mathrm{conv}(INT)$

Proof: $(DINT)$ separates into constraints for x (DN) and y (INT). The convexification only applies to y. $\square$

vii) Claim: $P(C) \cap \mathrm{conv}(DINB) = \underbrace{P(TC)}_{=P(T)\cap P(C)} \cap \mathrm{conv}(DINB)$

Proof: $(C,D) \Rightarrow (T)$. $\square$

viii) Claim: $z^I = z^{TI}$

Proof: $(C,D) \Rightarrow (T)$. $\square$

Claim: $z_D^{BT} = z_{TD}^B = z_T^D$

Proof: $P(D) \cap \mathrm{conv}(CIN) = \underbrace{P(TD)}_{(C,D)\Rightarrow(T)} \cap \mathrm{conv}(CIN) = P(TD) \cap \mathrm{conv}(\underbrace{CIN}_{(CIN)\Rightarrow(B)}B)$ $\square$

Claim: $z_{TD} = z^I$

Proof: $P(\cancel{TD}) \cap \mathrm{conv}(CINB) = P(TDCNB) = P(\cancel{TD}) \cap P(CNB)$

As in v)α) (without T), $\mathrm{conv}(BCIN) = P(BC) \cap \underbrace{\mathrm{conv}(IN)}_{=P(N)}$
$= P(BCN)$

Claim: $z_{BTC} = z^I$.

Proof: $P(BTC) \cap \underbrace{\mathrm{conv}(DIN)}_{x,y \text{ separate}} = P(BTCDN)$
$= P(D) \cap \mathrm{conv}(IN) = P(D) \cap P(N)$ $\square$

All other bounds are stronger than z^I and weaker than z_{TD} or z_{BTC}.

ix) Claim: $z^{BI} = z^{TBI}$

Proof: $(C,D) \Rightarrow (T)$. $\square$

Claim: $z^{BI} = z_{TC}^B$

Proof: $P(TCDN) = P(TC) \cap \underbrace{\mathrm{conv}(DIN)}_{=P(DN)}$. $\square$

The same argument yields the other equalities. $\square$